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OPTIMAL RECONSTRUCTIONS AND N-WIDTHS(U) TECHNIQ -
ISRAEL INST OF TECH HAIFA DEPT OF MATHEMATICS
A A MELKMAN ET AL. JUN 83 DAJA37-80-C-0234

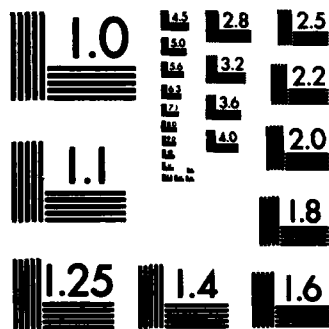
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Abstract

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Optimal Reconstructions and n-Widths

Final Technical Report

by

A. A. Melkman and A. Pinkus

June, 1983

United States Army

EUROPEAN RESEARCH OFFICE OF THE U. S. Army

London England

CONTRACT NUMBER DAJA37-8-C-0234

Technion Research and Development Foundation,
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. A132576	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Optimal Reconstructions and n-Widths		5. TYPE OF REPORT & PERIOD COVERED Final Report June 1983
7. AUTHOR(s) A. A. Melkman and A. Pinkus		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Technion - Israel Institute of Technology Department of Mathematics Haifa, Israel		8. CONTRACT OR GRANT NUMBER(s) DAJA37-8-C-0234
11. CONTROLLING OFFICE NAME AND ADDRESS USARDSG-UK Box 65 FPO NY 09510		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1T161102BH57-05
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE June 1983
		13. NUMBER OF PAGES Nine (9)
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release; Distribution Unlimited		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) n-widths, optimal reconstruction, Sobolev space, time- and band-limited signals.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This research has been concerned with the interrelated subjects of n-widths and optimal reconstruction. An attempt has been made to establish a mathematical framework within which a general theory may be developed. In addition, two important problems have been solved. The first of these deals with n-widths of Sobolev spaces, while the second is concerned with optimal reconstruction and n-widths of time- and band-limited signals.		

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Abstract

This research has been concerned with the interrelated subjects of n -widths and optimal reconstruction. An attempt has been made to establish a mathematical framework within which a general theory may be developed. In addition, two important problems have been solved. The first of these deals with n -widths of Sobolev spaces, while the second is concerned with optimal reconstruction and n -widths of time- and band-limited signals.

Keywords: n -widths, optimal reconstruction, Sobolev space, time- and band-limited signals.

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§1. Introduction

In Sections 2 and 3 we present the mathematical models of both n -widths and optimal reconstructions. Sections 4 and 5 are a discussion of work done by the principal investigator during the past two years in the area, and in particular with respect to n -widths of Sobolev spaces. In Section 6 we present some of the work recently done by the associate investigator on time- and band-limited signals.

§2. n-Widths : Background

Let X be a normed linear space and A a subset of X . For a given continuous linear rank n map P_n of X into itself, the quantity

$$\sup\{\|x - P_n x\| : x \in A\}$$

measures the degree to which A is approximated by the linear map P_n . The linear n -width of A in X is defined by

$$\delta_n(A; X) = \inf_{P_n} \sup_{x \in A} \|x - P_n x\|$$

where P_n runs over all continuous linear rank n maps. The linear n -width measures the extent to which A is approximable by such maps. If in place of linear approximants P_n , we consider best approximations from the range of P_n (in general a non-linear map) then we obtain, analogously, the Kolmogorov n -width of A in X as defined by

$$d_n(A; X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|,$$

where the left-most infimum runs over all n -dimensional subspaces X_n of X . The Kolmogorov n -width measures the degree to which n -dimensional subspaces can approximate A . It was, chronologically, the first n -width concept and was introduced by Kolmogorov [K] in 1936.

Two other n -width concepts have a less obvious motivation but deserve mention. The first of these is the Gel'fand n -width of A in X . It is given by

$$d^n(A; X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|,$$

where the L^n range over all closed subspaces of X of codimension at most n . One reason for generally introducing the Gel'fand n -width is due to a duality relationship which exists between it and the Kolmogorov n -width. More importantly, perhaps, is the fact that the Gel'fand n -width provides a lower bound on the problem of optimal reconstruction of elements of A based on n information data.

The Bernstein n -width of A in X is given by

$$b_n(A; X) = \sup_{X_{n+1}} \sup\{\lambda : \lambda S(X_{n+1}) \subseteq A\}$$

where $S(X_{n+1})$ is the unit ball in X_{n+1} , and X_{n+1} ranges over all subspaces of X of dimension at least $n+1$. The Bernstein n -width is important in that $\delta_n(A;X) \geq d_n(A;X)$, $d^n(A;X) \geq b_n(A;X)$. When strict inequalities occur then the determination of these quantities is often an insurmountable problem. In addition, the Bernstein n -width is also connected with the problem of optimal constants in Bernstein-Markov type inequalities.

One is generally interested in one of two problems for specific A and X . Either exactly determine the n -widths and identify optimal subspaces and operators, or determine the asymptotic values of the n -widths as $n \rightarrow \infty$.

The topics of n -widths in approximation theory (n -widths have also been successfully applied in other areas of mathematics) has been a vital subject for the last twenty or so years. One can grossly divide the main work on this subject into six areas. These are

- 1) basic properties
- 2) n -widths in Hilbert spaces
- 3) exact n -widths for integral operators (especially for Sobolev spaces)
- 4) n -widths and matrices
- 5) asymptotic estimates for n -widths of Sobolev spaces
- 6) n -widths of spaces of analytic functions.

§3. Optimal Reconstruction: Background

Optimal reconstruction, often called optimal recovery, is based, more or less, on the following model. Assume that X, Y, Z are normed linear spaces, and $U : X \rightarrow Z$ and $J : X \rightarrow Y$ are linear operators. The problem is to estimate Ux given the fact that $x \in A \subseteq X$, some a priori information Jx or perhaps the additional data $y \in Y$ which may be somewhat inaccurate as expressed by $\|Jx - y\| \leq \epsilon$. Given a map $S : Y \rightarrow Z$ one may take Sy as an estimate of Ux . In the worst case analysis, such a reconstruction S is normed by

$$E(S) = \sup\{\|Ux - Sy\| : x \in A, \|Jx - y\| \leq \epsilon\}.$$

The inherent error in the problem is $E = \inf E(S)$ and a recovery process S^* which achieves this infimum is called an optimal reconstruction.

When optimizing the information operator J , one is naturally led to an n -width problem. In particular to problems connected with Gel'fand n -widths (see e.g. Traub, Wozniakowski [TW]).

§4. General work

One of the primary objectives of this research was an attempt to establish a theoretical framework for the subject of n -widths in approximation theory. As such much effort originally went into general surveys of the subject. After completing two such surveys, the principal investigator was approached by Springer-Verlag Publishers. These surveys were considerably expanded and a monograph on n -widths in approximation theory should appear soon. A good deal of the monograph is original work, not only in its presentation, but also in many of the results. In addition, the principal investigator was invited to contribute a survey paper on n -widths at the conference in approximation theory held in College Station, Texas in January, 1983. The paper will appear in a book to be published by Academic Press.

§5. n -Widths in Sobolev Spaces

Let $W_p^{(r)}[0,1]$ denote the Sobolev space on $[0,1]$ defined by

$$W_p^{(r)}[0,1] = \{f: f^{(r-1)} \text{ abs. cont.}, f^{(r)} \in L^p\}$$

and set $B_p^{(r)} = \{f: f \in W_p^{(r)}[0,1], \|f^{(r)}\|_p \leq 1\}$, where $\|\cdot\|_p$ is the usual L^p -norm.

One of the major open problems in n -widths has been the determination of the n -widths of $B_p^{(r)}$ in L^q , $p, q \in [1, \infty]$. This problem had been solved for $p = q = 2$ by Kolmogorov [K] and Melkman, Micchelli [MM], for $p = q = \infty$ by Tichomirov [T], and for $p = \infty$, all q and $q = 1$, all p by Micchelli, Pinkus [MP]. On the basis of these results, various conjectures had been made concerning the n -widths of $B_p^{(r)}$ in L^q for $p > q$. The principal investigator has proved all these conjectures for $p = q \in [1, \infty]$. In particular

Theorem. Let $B_p^{(r)}$ be as defined above. Then

$$\delta_n(B_p^{(r)}; L^p) = d_n(B_p^{(r)}; L^p) = d^n(B_p^{(r)}; L^p) = b_n(B_p^{(r)}; L^p).$$

Furthermore, there exist for each p and r , two sets of points $\{\xi_i\}_{i=1}^{n-r}$ and $\{\eta_i\}_{i=1}^n$ such that

- 1) $X_n = \text{span}\{1, x, \dots, x^{r-1}, (x-\xi_1)_+^{r-1}, \dots, (x-\xi_{n-r})_+^{r-1}\}$
is optimal for $d_n(B_p^{(r)}; L^p)$ (i.e., splines of degree $r-1$ with $n-r$ fixed knots).
- 2) $L_n = \{f : f \in B_p^{(r)}, f(\eta_i) = 0, i = 1, \dots, n\}$ is optimal for $d_n(B_p^{(r)}; L^p)$.
- 3) P_n defined by interpolating $f \in B_p^{(r)}$ at the $\{\eta_i\}_{i=1}^n$ from X_n is optimal for $\delta_n(B_p^{(r)}; L^p)$.

§6. Time- and Band-Limited Signals

One aspect of the present investigations concerns the connection between "real world" phenomena and the mathematical models. This question was discussed by Slepian [S] in the 1974 Shannon Lecture. He suggested that there are certain constructs in a model, principal quantities, to which we attach physical significance. Others, secondary quantities, have no meaningful physical counterpart. In a "useful" model the principal quantities should be insensitive to small changes in the secondary quantities. We consider the example of time- and band-limited signals.

In modeling this class one is immediately confronted with two design decisions.

- a. what should be the mathematical class of functions?
- b. which norm should be imposed?

The problem with (a) is that the mathematical equivalent of band-limitedness, i.e., Fourier transform supported on a finite interval, implies immediately that the function is entire. Such a function cannot be time-limited. The way out of this paradox proposed by Slepian lies in the recognition that actual signals are de facto time- and band-limited because no measuring apparatus distinguishes them from zero outside a certain band-width and time-duration. Thus, in the model any mathematically reasonable class of functions is acceptable if it preserves the principal quantities, band-width and time-duration. In particular, one might choose to deal with strictly band-limited functions which are "immeasurably small" except during a specified time-duration. Such a function will model an actual signal only to within measurement accuracy.

As for (b), in the original series of papers by Landau, Pollack [LP], Slepian [S], and Melkman [M], the L^2 -norm was chosen. But this is not a foregone conclusion, and in fact a

cogent argument can be made for choosing the $\sup(L^\infty)$ -norm as was done by the associate investigator in the present work.

It is important that design decisions be buttressed by a verification that their major conclusions are independent of the particular decision, i.e., measurement accuracy and norm chosen. Two results obtained during the research program provide such verification.

Result 1. The number of independent signals of band-width σ and time-duration $2T$ (the dimensionality of the set) is $2\sigma T/\pi$ irrespective of the measurement accuracy and norm imposed (L^2 or L^∞).

Result 2. An optimal set of functions for the purpose of interpolation/approximation consists of sinc functions in both the L^2 - and L^∞ -norm.

It is conjectured that these results continue to hold in any L^p -norm.

In a paper to appear, these results are presented in a mathematical framework of optimal reconstruction and n -widths.

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